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A THERMOSYPHON MODEL WITH A VISCOELASTIC BINARY FLUID

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Abstract. In this work we consider a viscoelastic fluid with the same transfer law across the loop, as in previous works we add a solute to the fluid. For this binary fluid, we consider the thermodiffusion (also known as Soret effect) to obtain the well-posedness of the mathematical formulation of this thermosyphon model, which is a generalization of the previous models.

1. Introduction

In this work we consider the motion of the viscoelastic binary fluid inside a closed loop circuit (thermosyphon), when we consider a prescribed flux along the loop wall and the contribution of axial diffusion. This problem arises in engineering applications, where one deals with polymeric solutions that change the viscoelastic response of the solvent and can be segregated inside the fluid creating solute gradients.

Geometrically, a thermosyphon is a closed *pipe* containing a fluid and used, primarily, as a heat exchanger between different spatial locations. Their use is widespread in engineering so a deeper understanding of the effect of a prescribed external heat flux can be important to design external mechanisms that can control the flow within the thermosyphon.

Viscoelasticity [1], is produced by the internal composition of the fluid (that includes the solvent and the solute) that make them solid-like at low shear rates and water-like (Newtonian) at high shear rates (think, for instance, in ketchup or soap gels). Elasticity is the result of bond stretching along crystallographic planes in an ordered solid, whereas viscosity is the result of the diffusion of atoms or molecules inside an amorphous material.

Thermodiffusion is a phenomenon of temperature gradient [3, 8], observed in a mixture of two or more moving substances. The term "Soret effect" normally means thermodiffusion in liquids. Thus, inside a thermosyphon, besides the effect of temperature gradients, solute concentration gradients also trigger and drive natural convection inside the loop, hence sustaining the motion of the fluid [15].

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As it has been shown in previous works [5, 7, 10, 13, 14, 16, 18, 20, 21] there is a subtle coupling among gravity, natural convection and viscoelaticity, specially when cross-effects are present (such as thermal-solute concentration couplings or temperature dependent density).

The contributions in this article are:

• To generalize the system of equations (2.4) introduced in [23] governing this thermosyphon model of a viscoelastic binary fluid with Soret effect where, instead of leaving the motion be ambient-temperature driven as in [23], use an external heat flux prescription that can be used by an experimenter to control the fluid motion in the loop. This model, at different levels (the Soret effect, viscoelasticity or heat coupling), is a generalization of the previous models [5, 7, 10, 13, 14, 16, 18].

• To prove the well-posedness: existence and regularity of solutions for nonlinear coupled ODE/PDE system (2.4) arising in this new thermosyphon model.

2. Mathematical formulation of this prescribed heat flux model

For completeness, we include here some ideas of mathematical formulation for the present model, although the details are in [23] where we consider also a thermosyphon model with a viscoelstic binary fluid. The difference between this model and the recent work with a viscoelastc binary fluid [23], is that here, we consider a given function h to prescribed the heat flux at the wall of the loop instead of the Newton's linear cooling law. As shown in [10, 13, 18] this is not a trivial generalization and requires a detailed analysis.

In this thermosyphon model we study the motion (velocity v) of the viscoelastic binary fluid inside a closed loop. We note as the previous thermosyphon model for binary fluid, even for Newtonian fluid (like water) [5, 9, 10, 11, 12, 14, 16], together with the temperature T we study also de evolution of the solute concentration S; so in this kind of binary fluid we have and additional partial differential equation for the solute concentration, coupled with both the temperature and the velocity inside the loop.

Moreover, in this work we consider a viscoelastic fluid where the viscoelasticity is caused by the internal composition of the fluid (that includes the solvent and the solute). This kind of viscoelasticity fluid presents more complex dinamyc since the molecules responsible of the viscoelastic behavior (solute) can segregate inside the solvent producing concentration gradients sensible to thermal gradients (the Soret effect).

For small perturbations, the fluid behaves like an elastic solid with a characteristic frequency of resonance which, eventually, could be relevant. Here, we will approach this problem by studying the most essential feature of viscoelastic fluids: memory effects i.e. its behavior depends on the whole past history [4].

The simplest approach to viscoelasticity comes from the so-called Maxwell constitutive equation [17, 2]. Although this model is a great simplification, it has been proven valid even for complex fluids as blood, in which red cells change its behavior depending on their concentration or even the geometry of the vessel [19].

In this kind of fluid, both Newton's law of viscosity and Hooke's law of elasticity are generalized and complemented through an evolution equation for the stress tensor, $\tilde{\sigma}$. The stress tensor comes into play in the equation for the conservation of momentum:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\Big) = -\nabla p + \nabla\cdot\tilde{\sigma} + \rho g \tag{2.1}$$

where ρ is density of the material, p the hydrostatic pressure and g the acceleration due to gravity. We considere also the hypothesis of incompressibility (accurate enough for liquids).

For a Maxwellian fluid in a narrow section thermosyphon, the stress tensor is reduced to only one non-zero independent component, and evolves according to

$$\frac{\mu}{E}\frac{\partial\tilde{\sigma}}{\partial t}+\tilde{\sigma}=\mu\dot{\gamma}, \tag{2.2}$$

where μ is the fluid viscosity, E the Young's modulus and $\dot{\gamma}$ is the only non-zero component of the shear strain rate (or rate at which the fluid deforms).

We note that the equation (2.2) can be rewritten as

$$\sigma(t)=\sigma(0)+E\int_0^t e^{(E/\mu)(s-t)}\dot{\gamma}(s)ds \tag{2.3}$$

so, the so-called memory effect present in viscoelastic materials [17] is a way to rephrase the averaging effect shown in 2.3 over past times. This (weighted by an exponential) averaging can, for some parameters, remove the chaotic behavior inside the thermosyphon.

Under stationary flow, (2.2) reduces to Newton's law, $\tilde{\sigma}=\mu\dot{\gamma}$, and consequently equation (2.1) takes the form of the celebrated Navier-Stokes equation. On the contrary, for short times where *impulsive* behavior from rest can be expected, so $\mu\partial_t\tilde{\sigma}\gg E\tilde{\sigma}$, so equation (2.2) reduces to Hooke's law of elasticity, $\tilde{\sigma}=E\gamma$.

Following the same procedure as in [13], namely, averaging first (2.1)-(2.2), through the thermodiffusion section and, second, along its arclength, we arrive at a nonlinear coupled ODE/PDE system, where nonlinearity enters specifically in the equation for the velocity. In particular,

$$\begin{gathered}\varepsilon\frac{d^2v}{dt^2}+\frac{dv}{dt}+G(v)v=\oint(T-S)f\,dx,\quad v(0)=v_0,\frac{dv}{dt}(0)=w_0\\ \frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}=h(x)+\nu\frac{\partial^2 T}{\partial x^2},\quad T(0,x)=T_0(x)\\ \frac{\partial S}{\partial t}+v\frac{\partial S}{\partial x}=c\frac{\partial^2 S}{\partial x^2}-b\frac{\partial^2 T}{\partial x^2},\quad S(0,x)=S_0(x)\end{gathered} \tag{2.4}$$

where $h(x)$ is a given function which prescribed the heat flux at the wall of the loop as in [10, 13, 18], instead of the Newton's linear cooling law $h=k(T_a-T)$ as in [9, 11, 12, 20, 13, 23], where T_a is the (given) ambient temperature distribution. This is the difference between this model and the model in [23].

We consider the diffusion of temperature given by the term $\nu\frac{\partial^2 T}{\partial x^2}$, with thermal diffusion $\nu>0$ as in previous work.

The parameter ε in (2.4) is the (adimensional) time scale in which the transition from elastic to fluid-like occurs in the fluid. This forms an ODE/PDE system for the evolution of the velocity $v(t)$, the distribution of the temperature $T(t,x)$ of the fluid and the solute concentration $S(t,x)$ into the loop of (2.4). The equation for the solute concentration $S(t,x)$ is given by the Soret effect, thermodiffusion, where $c>0$ is the diffusion coefficient and $b>0$ is the Soret coefficient like in the previous models with this kind of binary fluids as [5, 9, 10, 11, 12]. Here $\oint=\int_0^1 dx$ denotes integration along the closed path of the circuit. We can make this identification if we consider only periodic functions (with period 1). The function f describes the geometry of the loop and the distribution of gravitational forces g [14, 21], with $\oint f = 0$.

We assume that $G(v)$, which specifies the friction law at the inner wall of the loop, is positive and bounded away from zero. This function has been usually taken to be $G(v) = G$, a positive constant for the linear friction case [14] (Stokes flow), or $G(v) = |v|$ for the quadratic law [7, 16], or even a rather general function given by $G(v) = \tilde{g}(Re)|v|$, where Re is the Reynolds number, $Re = \rho v L/\mu$. Here we will consider a general function of the velocity assumed to be large for large values of the velocity [20, 18]. The functions G, f, and h incorporate relevant physical constants of the model, such as the cross sectional area, D, the length of the loop, L, the Prandtl, Rayleigh, or Reynolds numbers, etc., see [20].

We consider G being a generic continuous function satisfying $G(v) \geq G_0 > 0$ and $H(r) = rG(r)$ being locally Lipschitz.

3. Well-posedness and boundedness: Existence and uniqueness of solutions

We will introduce some function spaces that will be used in the study of the existence of solutions of (2.4). Let $\Omega = (0, 1)$ and consider the spaces

$$\begin{gathered} L^2_{\rm per}(\Omega) = \big\{u \in L^2_{\rm loc}(\mathbb{R}), u(x+1) = u(x) \text{ a.e. } x \in \mathbb{R}\big\}, \\ H^m_{\rm per}(\Omega) = H^m_{\rm loc}(\mathbb{R}) \cap L^2_{\rm per}(\Omega) \end{gathered} \tag{3.1}$$

where $m \in \mathbb{N} \cup \{0\}$, and $u \in L^2_{\rm loc}(\mathbb{R})$ (or $H^m_{\rm loc}(\mathbb{R})$) if and only if for every open set $\omega \subset\subset \mathbb{R}$ one has $u \in L^2_{\rm loc}(\omega)$ (or $H^m_{\rm loc}(\omega)$, respectively). Finally, we consider functions with zero average, and we denote by

$$\dot{L}^2_{\rm per}(0,1) = \{u \in L^2_{\rm loc}(\mathbb{R}), u(x+1) = u(x) \text{ a.e., } \oint u = 0\}, \tag{3.2}$$

$$\dot{H}^m_{\rm per}(0,1) = H^m_{\rm loc}(\mathbb{R}) \cap \dot{L}^2_{\rm per}(0,1). \tag{3.3}$$

Note that the *dot* stand for functions with zero average, and it is not related to time derivatives of the functions.

In this section, we prove the existence and uniqueness of solutions of the thermosyphon model (2.4), with $f, h \in \dot{L}^2_{\rm per}(0,1)$, $T_0 \in \dot{H}^1_{\rm per}(0,1)$ and $S_0 \in \dot{L}^2_{\rm per}(0,1)$, where $\dot{L}^2_{\rm per}(0,1)$ and $\dot{H}^1_{\rm per}(0,1)$ are given by (3.3).

To choose the framework, we note that for $\nu > 0$, if we integrate the equation for the temperature along the loop, taking into account the periodicity of T, i.e., $\oint \frac{\partial T}{\partial x} = \oint \frac{\partial^2 T}{\partial x^2} = 0$, we have $\frac{d}{dt}(\oint T) = \oint h$, this is $\oint T = \oint T_0 + t \oint h$. Therefore, the temperature is unbounded, as $t \to \infty$, unless $\oint h = 0$. However, taking $\tau = T - \oint T$ and $h^* = h - \oint h$ reduces to the case $\oint T(t) = \oint T_0 = \oint h = 0$, since from the second equation of the system (2.4), τ satisfies the equation

$$\frac{\partial \tau}{\partial t} + v \frac{\partial \tau}{\partial x} = h(x) + \nu \frac{\partial^2 \tau}{\partial x^2}, \quad \tau(0,x) = \tau_0(x) = T_0 - \oint T_0.$$

Moreover, we integrate the equation for the solute concentration along the loop and taking into account the periodicity of S, i.e., $\oint \frac{\partial S}{\partial x} = \oint \frac{\partial^2 S}{\partial x^2} = 0$, we obtain $\frac{d}{dt}(\oint S) = 0$. As $\oint S$ is constant, it implies that the solute $\oint S = \oint S_0$ for all t.

We consider $\sigma = S - \oint S_0$, then from the third equation of the system (2.4), σ satisfies the equation (this σ is an auxiliary variable, not be confused with the stress)

$$\frac{\partial \sigma}{\partial t} + v\frac{\partial \sigma}{\partial x} = c\frac{\partial^2 \sigma}{\partial x^2} - b\frac{\partial^2 \tau}{\partial x^2}, \quad \sigma(0,x) = \sigma_0(x) = S_0 - \oint S_0.$$

Since $\oint f = 0$, we have $\oint (T-S)f = \oint (\tau - \sigma) f$ and the equations for v is

$$\varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint (\tau - \sigma) f, \quad v(0) = v_0, \quad \frac{dv}{dt}(0) = w_0.$$

Therefore, we obtain (v, τ, σ) satisfying the system (2.4) with τ_0, σ_0 replacing T_0, S_0 respectively and $\oint f = \oint \tau_0 = \oint \sigma_0 = 0$ and $\oint T(t) = \oint S(t) = 0$ for all $t \geq 0$. Therefore, hereafter we consider all the functions of the system (2.4) to have zero average.

Also, if $\nu, c > 0$ the operators $\nu A = -\nu \frac{\partial^2}{\partial x^2}$ and $cA = -c\frac{\partial^2}{\partial x^2}$, together with periodic boundary conditions, are unbounded, self-adjoint operators with compact resolvent in $L^2_{\rm per}(0,1)$, that are positive when restricted to the space of zero average functions in $\dot{L}^2_{\rm per}(0,1)$. Hence, the equation for the temperature T and the equation for the solute concentration S in (2.4) are of parabolic type for $\nu, c > 0$.

We write the system (2.4) as the following evolution system for acceleration, velocity, temperature and solute concentration:

$$\begin{gathered}
\frac{dw}{dt} + \frac{1}{\varepsilon} w = -\frac{1}{\varepsilon} G(v)v + \frac{1}{\varepsilon}\oint (T-S)f, \quad w(0) = w_0. \\
\frac{dv}{dt} = w, \quad v(0) = v_0. \\
\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} - \nu \frac{\partial^2 T}{\partial x^2} = h, \quad T(0,x) = T_0(x), \\
\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial x} = c\frac{\partial^2 S}{\partial x^2} - b\frac{\partial^2 T}{\partial x^2}, \quad S(0,x) = S_0(x).
\end{gathered} \tag{3.4}$$

That is,

$$\frac{d}{dt}\begin{pmatrix} w \\ v \\ T \\ S \end{pmatrix} + \begin{pmatrix} 1/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & -c\frac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} w \\ v \\ T \\ S \end{pmatrix} = \begin{pmatrix} F_1(w,v,T,S) \\ F_2(w,v,T,S) \\ F_3(w,v,T,S) \\ F_4(w,v,T,S) \end{pmatrix} \tag{3.5}$$

with

$$\begin{gathered}
F_1(w,v,T,S) = -\frac{1}{\varepsilon} G(v)v + \frac{1}{\varepsilon}\oint (T-S)f, \quad F_2(w,v,T,S) = w, \\
F_3(w,v,T,S) = -v\frac{\partial T}{\partial x} + h, \quad F_4(w,v,T,S) = -v\frac{\partial S}{\partial x} - b\frac{\partial^2 T}{\partial x^2},
\end{gathered}$$

and initial data

$$\begin{pmatrix} w \\ v \\ T \\ S \end{pmatrix}(0) = \begin{pmatrix} w_0 \\ v_0 \\ T_0 \\ S_0 \end{pmatrix}.$$

The operator

$$B = \begin{pmatrix} 1/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & -c\frac{\partial^2}{\partial x^2} \end{pmatrix}$$

is a sectorial operator in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$ with domain $D(B) = \mathbb{R}^2 \times \dot{H}^3_{\rm per}(0,1) \times \dot{H}^2_{\rm per}(0,1)$ and has compact resolvent, see (3.3).

Using the result and techniques about sectorial operator in [6] to prove the existence of solutions of the system, we have the following result.

Theorem 3.1. *We assume that $H(r) = rG(r)$ is locally Lipschitz, and that $f, h \in \dot{L}^2_{\rm per}(0,1)$, with $G(v) \geq G_0 > 0$. Then, given $(w_0, v_0, T_0, S_0) \in \mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$, there exists a unique solution of (2.4) satisfying*

$$(w, v, T, S) \in C([0,\infty), \mathcal{Y}) \cap C(0, \infty, \mathbb{R}^2 \times \dot{H}^3_{\rm per}(0,1) \times \dot{H}^2_{\rm per}(0,1)),$$
$$\Big(\frac{dw}{dt}, \frac{dv}{dt}, \frac{\partial T}{\partial t}, \frac{\partial S}{\partial t}\Big) \in C(0, \infty, \mathbb{R}^2 \times \dot{H}^{3-\delta}_{\rm per}(0,1) \times \dot{H}^{2-\delta}_{\rm per}(0,1)),$$

for every $\delta > 0$. In particular, (3.4) defines a nonlinear semigroup, $S^*(t)$ in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$, with $S^*(t)(w_0, v_0, T_0, S_0) = (w(t), v(t), T(t,x), S(t,x))$.

Proof. **Step (i)** We prove the local existence and regularity. This follows easily from the variation of constants formula of [6]. To prove this, we write the system as (3.5), and we have

$$U_t + BU = F(U), \quad \text{with } U = \begin{pmatrix} w \\ v \\ T \\ S \end{pmatrix},$$
$$B = \begin{pmatrix} 1/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & -c\frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

where the operator B is a sectorial operator in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$ with domain $D(B) = \mathbb{R}^2 \times \dot{H}^3_{\rm per}(0,1) \times \dot{H}^2_{\rm per}(0,1)$ and has compact resolvent. In this context, the operator $A = -\frac{\partial^2}{\partial x^2}$ must be understood in the variational sense, i.e., for every $T, \varphi \in \dot{H}^1_{\rm per}(0,1)$,

$$\langle A(T), \varphi\rangle = \oint \frac{\partial T}{\partial x}\frac{\partial \varphi}{\partial x}$$

and $\dot{L}^2_{\rm per}(0,1)$ coincides with the fractional space of exponent $\frac{1}{2}$ as in [6]. We denote $\dot{H}^{-1}_{\rm per}(0,1)$ as the dual space and $\|\cdot\|$ the norm on the space $\dot{L}^2_{\rm per}(0,1)$. If we prove that the nonlinearity $F : \mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1) \mapsto \mathcal{Y}^{-\frac{1}{2}} = \mathbb{R}^2 \times \dot{L}^2_{\rm per}(0,1) \times \dot{H}^{-1}_{\rm per}(0,1)$ is well defined, Lipschitz and bounded on bounded sets, we obtain the local existence for the initial data in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$.

Using the fact that $H(v) = G(v)v$ is locally Lipschitz together with $f, h \in \dot{L}^2_{\rm per}(0,1)$, we will prove the nonlinear terms,

$$F_1(w, v, T, S) = -\frac{1}{\varepsilon}G(v)v + \frac{1}{\varepsilon}\oint (T - S)f, \quad F_2(w, v, T, S) = w,$$
$$F_3(w, v, T, S) = -v\frac{\partial T}{\partial x} + h, \quad F_4(w, v, T, S) = -v\frac{\partial S}{\partial x} - b\frac{\partial^2 T}{\partial x^2}$$

satisfy $F_1 : \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1) \mapsto \mathbb{R}$, $F_2 : \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1) \mapsto \mathbb{R}$, $F_3 : \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1) \mapsto \dot{L}^2_{\rm per}(0,1)$ and $F_4 : \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1) \mapsto \dot{H}^{-1}_{\rm per}(0,1)$; that is, $F : \mathcal{Y} \mapsto \mathcal{Y}^{-\frac{1}{2}}$ is well defined, Lipschitz and bounded on bounded sets.

Using the techniques of variation of constants formula [6], we obtain the unique local solution $(w, v, T, S) \in C([0, t^*], \mathcal{Y})$ (with a suitable $t^* > 0$) of (3.4), which are given by

$$w(t) = w_0 e^{-\frac{1}{\varepsilon}t} - \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-r)} H(r)dr + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(t-r)} \oint (T-S)f(r)dr \tag{3.6}$$

with $H(r) = G(v(r))v(r)$.

$$v(t) = v_0 + \int_0^t w(r)dr, \tag{3.7}$$

$$T(t,x) = e^{-\nu At}T_0(x) + \int_0^t e^{-\nu A(t-r)}h(x)]dr - \int_0^t e^{-\nu A(t-r)}v(r)\frac{\partial T(r,x)}{\partial x}dr, \tag{3.8}$$

$$S(t,x) = e^{-cAt}S_0(x) + \int_0^t e^{-cA(t-r)}[-v(r)\frac{\partial S}{\partial x}(r) - b\frac{\partial^2 T}{\partial x^2}(r)]dr. \tag{3.9}$$

where $(w, v, T, S) \in C([0, t^*], \mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1))$ and using again the results of [6], (smoothing effect of the equations together with bootstrapping method), we obtain the regularity of solutions.

Step (ii) To prove the global existence, we must show that the solutions are bounded in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$ for finite time intervals and using the nonlinearity of F, maps bounded on bounded sets, we conclude.

To obtain the norm of T is bounded in finite time, we multiply the equation for the temperature by T in $\dot{L}^2_{\rm per}(0,1)$. Then integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \nu\|\frac{\partial T}{\partial x}\|^2 = \oint hT dx$$

since $\oint T\frac{\partial T}{\partial x} = \frac{1}{2}\oint \frac{\partial}{\partial x}(T^2) = 0$.

Using Cauchy-Schwartz and Young inequality and then the Poincaré inequality for functions of zero average, since $\oint T = 0$, together with π^2 is the first nonzero eigenvalue of $A = -\frac{\partial^2}{\partial x^2}$ in $\dot{L}^2_{\rm per}(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \nu\pi^2\|T\|^2 \le C_\delta\|h\|^2 + \delta\|T\|^2,$$

for every $\delta > 0$ with $C_\delta = 1/(4\delta)$. Thus, taking $\delta = \nu\pi^2/2$, $C_\delta = 1/(2\nu\pi^2)$, we obtain

$$\frac{d}{dt}\|T\|^2 + \nu\pi^2\|T\|^2 \le \frac{\|h\|^2}{\nu\pi^2}, \tag{3.10}$$

and we conclude that the norm of T in $\dot{L}^2_{\rm per}(0,1)$ remains bounded in finite time.

Now, we prove that the norm $\|\frac{\partial T}{\partial x}\|$ remains bounded in finite time intervals. For this, multiply the third equation of (3.4) by $-\frac{\partial^2 T}{\partial x^2}$ in $\dot{L}^2_{\rm per}(0,1)$. Integrating by parts, applying the Young inequality and taking into account that

$$\oint \frac{\partial T}{\partial x}\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}\oint \frac{\partial(\partial T/\partial x)^2}{\partial x} = 0,$$

since $\partial T/\partial x$ is periodic, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial T}{\partial x}\|^2 + \nu\|\frac{\partial^2 T}{\partial x^2}\|^2 \le C_\delta\|h\|^2 + \delta\|\frac{\partial^2 T}{\partial x^2}\|^2$$

for every $\delta > 0$ with $C_\delta = 1/(4\delta)$. Thus, taking $\delta = \nu/2$, and applying the Poincaré inequality for functions with zero average we obtain

$$\frac{d}{dt}\|\frac{\partial T}{\partial x}\|^2 + \nu\pi^2\|\frac{\partial T}{\partial x}\|^2 \le \frac{\|h\|^2}{\nu}, \tag{3.11}$$

since π^2 is the first nonzero eigenvalue of A in $\dot{L}^2_{\rm per}(0,1)$.

Thus we show that the norm of T in $\dot{H}^1_{\rm per}(0,1)$ remains bounded in finite time.

Finally, we show that the norm of S in $\dot{L}^2_{\rm per}(0,1)$ does not blow-up in finite time. Multiplying the fourth equation of (3.4) by S, integrating by parts, applying the Young inequality and again taking into account that

$$\oint S\frac{\partial S}{\partial x} = \frac{1}{2}\oint \frac{\partial S^2}{\partial x} = 0,$$

since S is periodic, we obtain

$$\frac{1}{2}\frac{d}{dt}\|S\|^2 + (c-\delta)\|\frac{\partial S}{\partial x}\|^2 \le b^2 C_\delta\|\frac{\partial T}{\partial x}\|^2 \tag{3.12}$$

for every $\delta > 0$ with $C_\delta = 1/(4\delta)$. Thus, taking $\delta = c/2$, together with the Poincaré inequality for functions with zero average, we obtain

$$\frac{d}{dt}\|S\|^2 + c\pi^2\|S\|^2 \le \frac{b^2}{c}\|\frac{\partial T}{\partial x}\|^2 \le k_1 \tag{3.13}$$

with $k_1 > 0$. Therefore $\|S(t)\|$ remains bounded in finite time. Since $\|T\|$ and $\|S\|$ are bounded in finite time, imply that $|w(t)|, |v(t)|$ remain also bounded in finite time. Hence we have a global solution in the nonlinear semigroup in $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^1_{\rm per}(0,1) \times \dot{L}^2_{\rm per}(0,1)$. $\square$

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